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Explicit solutions of integrable lattices

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Abstract

Explicit examples of quadrilateral lattices and their integrable reductions of pseudo-circular, symmetric and pseudo-Egorov types are presented. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper focuses its attention on the integrable aspects of discrete geometry [2]. Our main result is the construction of explicit families of quadrilateral, pseudo-circular, symmetric and pseudo-Egorov lattices by applying particular fundamental transformations [9,12] to the Cartesian lattice. This particular choice is suggested by previous papers [6,8,13,15,16] in which the Cauchy propagator [19] was extensively used in the study of integrable lattices and nets. In fact, our matrix function $D(z, z')$ introduced below can be understood as the Cauchy propagator of a particular Cartesian lattice and our fundamental transformations as dressing transformations of it. The advantage of this $D(z, z')$ compared to that used in, for example, [6,8] is that the reductions follow the same patron as in the continuous case and the $\bar{\partial}$ reduction theory simplifies (private communication by L. Bogdanov).

As the solutions obtained in this paper are produced by applying fundamental transformations, one should expect some N-dimensional discrete integration in order to find the transformation potentials. However, this is not the case, and only complex integration is

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used. The situation is even more interesting for some examples of pseudo-circular lattices given in terms of arbitrary discrete functions (Fourier coefficients of arbitrary measures) in which no integration is needed at all. Finally, the symmetric pseudo-circular lattices given here appear when particular limits—in which some singular terms have cancelled—are taken.

Now, we shortly review some basic aspects of integrable lattices and their fundamental transformations. The layout of the paper is as follows. Section 2 is devoted to present our ample families of explicit quadrilateral lattices and in Section 3, we characterize those among these families that reduce.

1.1. Quadrilateral lattices

Among the N-dimensional lattices $\boldsymbol{x}:\mathbb{Z}^N\to\mathbb{R}^N$ there is a distinguished class for which the elementary quadrilaterals are planar [7,9,18]. The planarity condition can be expressed by the following linear equation for suitably renormalized tangent vectors $\mathfrak{C}_i(\boldsymbol{n})\in\mathbb{R}^N$:

$$\Delta_j\mathfrak{C}_i=(T_jQ_{ij})\mathfrak{C}_j,\quad i,j=1,\dots,N,\ \ i\neq j, \tag{1}$$

being its compatibility conditions the following discrete Darboux equations [3]:

$$\Delta_kQ_{ij}=(T_kQ_{ik})Q_{kj},\quad i,j\ \text{ and }\ k\ \text{ different}.$$

The points $\boldsymbol{x}$ of the lattice can be found by means of discrete integration of

$$\Delta_i\boldsymbol{x}=(T_iH_i)\mathfrak{C}_i,\quad i=1,\dots,N,$$

where H_i are solutions of the equations

$$\Delta_iH_j=Q_{ij}T_iH_i,\quad i,j=1,\dots,N,\ \ i\neq j. \tag{2}$$

In the above formulas, T_i is the translation operator in the discrete variable n_i:

$$T_if(n_1,\dots,n_i,\dots,n_N)=f(n_1,\dots,n_i+1,\dots,n_N),$$

and $\Delta_i=T_i-1$ is the corresponding partial difference operator. As was explained in [8] there is an equivalent description in terms of backward geometrical objects, $\tilde{\mathfrak{C}}_i$, $\tilde{H}_i$, $\tilde{Q}_{ij}$ which satisfy

$$\Delta_i\tilde{\mathfrak{C}}_j=Q_{ij}T_i\tilde{\mathfrak{C}}_i,\qquad \Delta_j\tilde{H}_i=(T_j\tilde{Q}_{ij})\tilde{H}_j.$$

There exists first potentials $\rho_i, i=1,\dots,N$ [8] such that

$$\mathfrak{C}_i=-\rho_iT_i\tilde{\mathfrak{C}}_i,\qquad H_i=-\frac{1}{\rho_i}\tilde{H}_i,\qquad \rho_jT_j\tilde{Q}_{ij}=\rho_iT_iQ_{ji}.$$

The discrete vectorial fundamental transformation [9,12] is given by

$$\begin{aligned}
&Q'_{ij}=Q_{ij}-\boldsymbol{\Phi}^*_j\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}\boldsymbol{\Phi}_i,\quad i,j=1,\dots,N,\ \ i\neq j,\\
&H'_i=H_i-\boldsymbol{\Phi}^*_i\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}\Omega(\boldsymbol{\Phi},H),\qquad \mathfrak{C}'_i=\mathfrak{C}_i-\Omega(\mathfrak{C},\boldsymbol{\Phi}^*)\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}\boldsymbol{\Phi}_i,\\
&i=1,\dots,N,\qquad \boldsymbol{x}'=\boldsymbol{x}-\Omega(\mathfrak{C},\boldsymbol{\Phi}^*)\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)^{-1}\Omega(\boldsymbol{\Phi},H).
\end{aligned}$$

These are data for a new quadrilateral lattice $\boldsymbol{x}'$ provided $\boldsymbol{\Phi}_i \in V$, where V being a linear space, and $\boldsymbol{\Phi}_i^* \in V^*$, with V^* the dual of V, are solutions of (1) and (2), respectively. The linear operator $\Omega(\boldsymbol{\zeta}, \boldsymbol{\xi}^*) : W \to V$ is defined by the compatible system of equations:

$$\Delta_i \Omega(\boldsymbol{\zeta}, \boldsymbol{\xi}^*) = \boldsymbol{\zeta}_i \otimes (T_i \boldsymbol{\xi}_i^*), \quad i = 1, \dots, N. \tag{3}$$

The first potentials transform according to [14]

$$\rho_i' = \rho_i (1 + (T_i \boldsymbol{\Phi}_i^*) \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^{-1} \boldsymbol{\Phi}_i).$$

1.2. Reduced lattices

Quadrilateral lattices $\boldsymbol{x} : \mathbb{Z}^N \to \mathbb{R}^N$ for which each quadrilateral is inscribed in a circle are called circular or cyclid lattices [1,4,6,10,17]. It can be shown that the constraint

$$\mathfrak{C}_i \cdot T_i(\mathfrak{C}_j) + \mathfrak{C}_j \cdot T_j(\mathfrak{C}_i) = 0, \quad i \neq j \tag{4}$$

for the tangent vectors is equivalent to the requirement that lattice is circular. The first potentials for the circular lattices satisfy $\rho_i = \|\mathfrak{C}\|_i^2$ [8]. In [8], the symmetric and Egorov lattices were introduced—the Egorov lattice was also introduced by Schief. The symmetric lattice appears when backward and forward rotation coefficients are the same, which can be casted in the condition

$$(T_i Q_{ji})(T_j Q_{kj})(T_k Q_{ik}) = (T_j Q_{ij})(T_i Q_{ki})(T_k Q_{jk}), \quad i, j \text{ and } k \text{ different}.$$

In this case the first potentials satisfy $\rho_j T_j Q_{ij} = \rho_i T_i Q_{ji}$. A circular, symmetric and diagonal invariant lattice is called Egorov lattice, it was proven that Egorov lattices are characterized by

$$\mathfrak{C}_i \cdot T_i(\mathfrak{C}_j) = 0, \quad i \neq j. \tag{5}$$

Finally, in [13] pseudo-circular and pseudo-Egorov lattices in pseudo-Euclidean space $R_{p,q}$, $p+q = N$, have been introduced. Here, we have a non-degenerate symmetric bilinear form

$$\boldsymbol{X} \cdot \tilde{\boldsymbol{X}} := \sum_{i=1}^{N} \epsilon_i X_i \tilde{X}_i \quad \text{with } \epsilon_i := \begin{cases} 1, & i = 1, \dots, p, \\ -1, & i = p+1, \dots, p+q, \end{cases}$$

which can be written as $\boldsymbol{X} \cdot \tilde{\boldsymbol{X}} = (X_1, \dots, X_N) I_{p,q} \begin{pmatrix} \tilde{X}_1 \\ \vdots \\ \tilde{X}_N \end{pmatrix}$ with

$$I_{p,q} := \operatorname{diag}(\epsilon_1, \dots, \epsilon_N).$$

The pseudo-circular and pseudo-Egorov lattices are defined as in (4) and (5) but replacing the Euclidean scalar product by the pseudo-Euclidean scalar product just introduced.

When the data defining the fundamental transformation satisfy

$$\boldsymbol{\Phi}_i = (\Omega(\mathfrak{C}, \boldsymbol{\Phi}^*) + T_i \Omega(\mathfrak{C}, \boldsymbol{\Phi}^*))^\tau \mathfrak{C}_i, \quad i = 1, \dots, N,$$
$$\Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*) + \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^\tau = 2\Omega(\mathfrak{C}, \boldsymbol{\Phi}^*)^\tau \Omega(\mathfrak{C}, \boldsymbol{\Phi}^*),$$

where $A^\tau := I_{p,q} A^{\mathrm{t}} I_{p,q}$, the transformation preserves the pseudo-circular reduction [5,10, 11,13]. Finally, in [14] we have shown that if the transformation data fulfil

$$\boldsymbol{\Phi}_i = \rho_i T_i \boldsymbol{\Phi}_i^{*\mathrm{t}}, \qquad \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*) = \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^{\mathrm{t}},$$

the symmetric reduction is preserved [13]. Moreover, if

$$\boldsymbol{\Phi}_i = \epsilon_i \rho_i I_{p,q} T_i \boldsymbol{\Phi}_i^{*\mathrm{t}} = (\Omega(\mathfrak{C}, \boldsymbol{\Phi}^*) + T_i \Omega(\mathfrak{C}, \boldsymbol{\Phi}^*))^\tau \mathfrak{C}_i,$$
$$\Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*) = \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^\tau = \Omega(\mathfrak{C}, \boldsymbol{\Phi}^*)^\tau \Omega(\mathfrak{C}, \boldsymbol{\Phi}^*),$$

then the pseudo-Egorov reduction is also preserved.

2. Quadrilateral lattices

In this section, we give exact and explicit examples of ample families of quadrilateral lattices. For that aim we dress the Cartesian background with specific fundamental transformation. Finally, we describe the new quadrilateral lattice obtained.

2.1. Cartesian lattice

Our departing point is a Cartesian lattice characterized by the following rotation coefficients, Lamé coefficients, re-normalized tangent vectors and points of the lattice

$$Q_{ij}(\boldsymbol{n}) := 0, \quad i, j = 1, \dots, N, \ \ i \neq j, \qquad H_i(\boldsymbol{n}) := -\frac{p_i - q_i}{p_i}\left(\frac{q_i}{p_i}\right)^{n_i - 1},$$
$$x\mathfrak{C}_i(\boldsymbol{n}) := -\frac{1}{q_i}\left(\frac{q_i}{p_i}\right)^{n_i} \boldsymbol{e}_i, \quad i = 1, \dots, N, \qquad \boldsymbol{x}(\boldsymbol{n}) := \sum_{i=1}^{N} \frac{p_i - q_i}{p_i q_i} n_i \boldsymbol{e}_i,$$

where $\{p_i, q_i\}_{i=1}^N \subset \mathbb{C}$ are complex numbers with $p_i \neq q_i$; $\boldsymbol{n} = \sum_{i=1}^N n_i \boldsymbol{e}_i$, $n_i \in \mathbb{Z}$; $\{\boldsymbol{e}_i\}_{i=1}^N$ is the canonical basis in $\mathbb{R}^N$.

2.2. Transformation data and transformation potentials

The vectorial fundamental transformation [9,12] that we are going to perform is generated by the following transformation data:

$$\boldsymbol{\Phi}_i := \begin{pmatrix} \boldsymbol{\Phi}_{i1} \\ \vdots \\ \boldsymbol{\Phi}_{im} \end{pmatrix}, \qquad \boldsymbol{\Phi}_i^* := (\boldsymbol{\Phi}_{i1}^*, \dots, \boldsymbol{\Phi}_{im}^*), \tag{6}$$

where

$$\boldsymbol{\Phi}_{ik}(\boldsymbol{n}) := \int_{\mathbb{C}} f_k(z) \left(\frac{z - p_i}{z - q_i}\right)^{n_i} \frac{1}{z - q_i} \, \mathrm{d}^2 z \, \boldsymbol{e}_i,$$
$$\boldsymbol{\Phi}_{ik}^*(\boldsymbol{n}) := \int_{\mathbb{C}} \boldsymbol{e}^i \frac{p_i - q_i}{z - p_i} \left(\frac{z - q_i}{z - p_i}\right)^{n_i - 1} g_k(z) \, \mathrm{d}^2 z,$$

and $\{f_k(z), g_k(z)\}_{k=1}^m$ is a set of m $N \times N$ matrix distributions.

Observe that as

$$\Delta_i \mathbf{\Phi}_j = 0, \qquad \Delta_i \mathbf{\Phi}_j^* = 0, \quad j \neq i,$$

(1) and (2) are fulfilled and $\mathbf{\Phi}_i$, $\mathbf{\Phi}_i^*$, $i = 1, \dots, N$, are suitable transformation data for the Cartesian background. Now, we define the diagonal matrix

$$D(z, z', \boldsymbol{n}) := \frac{1}{z - z'} \sum_{i=1}^{N} \left[\frac{(z' - q_i)(z - p_i)}{(z' - p_i)(z - q_i)} \right]^{n_i} P_i,$$

where $P_i \boldsymbol{e}_j = \delta_{ij} \boldsymbol{e}_i$, which has the following important property:

$$\Delta_i D(z, z') = \frac{p_i - q_i}{(z' - p_i)(z - q_i)} \left[\frac{(z' - q_i)(z - p_i)}{(z' - p_i)(z - q_i)} \right]^{n_i} P_i. \tag{7}$$

In what follows this function will play a central role and it could be understood as the Cauchy propagator of the Cartesian background. With this matrix at hand we introduce

$$\Omega_{kl}(\mathbf{\Phi}, \mathbf{\Phi}^*) := C_{kl} + \int_{\mathbb{C}\times\mathbb{C}} f_k(z) D(z, z') g_l(z') \, \mathrm{d}^2 z \, \mathrm{d}^2 z', \tag{8}$$

$$\Omega_k(\mathbf{\Phi}, H) := \int_{\mathbb{C}} f_k(z) D(z, 0) \, \mathrm{d}^2 z \sum_{i=1}^{N} \boldsymbol{e}_i, \tag{9}$$

$$\Omega_k(\mathfrak{C}, \mathbf{\Phi}^*) := \int_{\mathbb{C}} D(0, z) g_k(z) \, \mathrm{d}^2 z, \tag{10}$$

where C_{kl} is an arbitrary $N \times N$ matrix. Now, we shall prove that these matrix functions are transformation potentials.

Proposition 1. *The transformation potentials just introduced fulfil the following relations*:

$$\Delta_i \Omega_{kl}(\mathbf{\Phi}, \mathbf{\Phi}^*) = \mathbf{\Phi}_{ik} \otimes T_i \mathbf{\Phi}_{il}^*, \tag{11}$$

$$\Delta_i \Omega_k(\mathbf{\Phi}, H) = \mathbf{\Phi}_{ik} T_i H_i, \tag{12}$$

$$\Delta_i \Omega_k(\mathfrak{C}, \mathbf{\Phi}^*) = \mathfrak{C}_i \otimes T_i \mathbf{\Phi}_{ik}^*. \tag{13}$$

Proof. For (11), we just apply Δ_i to (8) to get

$$\begin{aligned}
\Delta_i \Omega_{kl}(\mathbf{\Phi}, \mathbf{\Phi}^*) &= \int_{\mathbb{C}\times\mathbb{C}} f_k(z)(\Delta_i D(z, z', \boldsymbol{n})) g_l(z') \, \mathrm{d}^2 z \, \mathrm{d}^2 z' \\
&= \int_{\mathbb{C}\times\mathbb{C}} f_k(z) \frac{p_i - q_i}{(z' - p_i)(z - q_i)} \left[\frac{(z' - q_i)(z - p_i)}{(z' - p_i)(z - q_i)} \right]^{n_i} P_i g_l(z') \, \mathrm{d}^2 z \, \mathrm{d}^2 z' \\
&= \int_{\mathbb{C}} f_k(z) \left[\frac{z - p_i}{z - q_i} \right]^{n_i} \frac{1}{z - q_i} \boldsymbol{e}_i \, \mathrm{d}^2 z \otimes \int_{\mathbb{C}} \boldsymbol{e}^i \frac{p_i - q_i}{z' - p_i} \\
&\quad \times \left[\frac{z' - q_i}{z' - p_i} \right]^{n_i} g_l(z') \, \mathrm{d}^2 z' = \mathbf{\Phi}_{ik} \otimes T_i \mathbf{\Phi}_{il}^*.
\end{aligned}$$

The relation (12) follows by applying the difference operator to the definition (9), in doing so we get

$$\begin{aligned}\Delta_i \Omega_k(\boldsymbol{\Phi}, H) &= \int_{\mathbb{C}} f_k(z)(\Delta_i D(z,0))\,\mathrm{d}^2 z \sum_{j=1}^{N} \boldsymbol{e}_j \\ &= \int_{\mathbb{C}} f_k(z) \frac{p_i - q_i}{-p_i(z-q_i)} \left[\frac{q_i(z-p_i)}{p_i(z-q_i)}\right]^{n_i} \mathrm{d}^2 z\, \boldsymbol{e}_i \\ &= \left(\int_{\mathbb{C}} f_k(z) \left[\frac{z-p_i}{z-q_i}\right]^{n_i} \frac{1}{z-q_i} \boldsymbol{e}_i\, \mathrm{d}^2 z\right)\left(-\frac{p_i - q_i}{p_i}\left[\frac{q_i}{p_i}\right]^{n_i}\right) = \boldsymbol{\Phi}_{ik} T_i H_i.\end{aligned}$$

Finally, a similar reasoning gives (13)

$$\begin{aligned}\Delta_i \Omega_k(\mathfrak{C}, \boldsymbol{\Phi}^*) &= \int_{\mathbb{C}} (\Delta_i D(0,z)) g_k(z)\,\mathrm{d}^2 z \\ &= \int_{\mathbb{C}} \frac{p_i - q_i}{-(z-p_i)q_i}\left[\frac{(z-q_i)p_i}{(z-p_i)q_i}\right]^{n_i} P_i g_k(z)\,\mathrm{d}^2 z \\ &= \left(-\frac{1}{p_i}\left[\frac{p_i}{q_i}\right]^{n_i} \boldsymbol{e}_i\right) \otimes \left(\int_{\mathbb{C}} \boldsymbol{e}_i \frac{p_i - q_i}{z - q_i}\left[\frac{z-q_i}{z-p_i}\right]^{n_i} g_k(z)\,\mathrm{d}^2 z\right) \\ &= \mathfrak{C}_i \otimes T_i \boldsymbol{\Phi}^*_{ik}. \qquad \square\end{aligned}$$

We now introduce the notation

$$\Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*) := \begin{pmatrix} \Omega_{11}(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*) & \cdots & \Omega_{1m}(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*) \\ \vdots & & \vdots \\ \Omega_{m1}(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*) & \cdots & \Omega_{mm}(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*) \end{pmatrix},$$

$$\Omega(\boldsymbol{\Phi}, H) := \begin{pmatrix} \Omega_1(\boldsymbol{\Phi}, H) \\ \vdots \\ \Omega_m(\boldsymbol{\Phi}, H) \end{pmatrix}, \qquad \Omega(\mathfrak{C}, \boldsymbol{\Phi}^*) := (\Omega_1(\mathfrak{C}, \boldsymbol{\Phi}^*), \ldots, \Omega_m(\mathfrak{C}, \boldsymbol{\Phi}^*)),$$

so that we can rewrite Proposition 1 as

$$\begin{aligned}&\Delta_i \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*) = \boldsymbol{\Phi}_i \otimes T_i \boldsymbol{\Phi}^*_i, \qquad \Delta_i \Omega(\boldsymbol{\Phi}, H) = \boldsymbol{\Phi}_i T_i H_i, \\ &\Delta_i \Omega(\mathfrak{C}, \boldsymbol{\Phi}^*) = \mathfrak{C}_i \otimes T_i \boldsymbol{\Phi}^*_i,\end{aligned}$$

and conclude, following [9,12], that $\boldsymbol{\Phi}, \boldsymbol{\Phi}^*$ generate a vectorial fundamental transformation on the Cartesian background, with transformation potentials as described. Thus, a new quadrilateral lattice is given by

$$\begin{aligned}&Q'_{ij} = -\boldsymbol{\Phi}^*_j \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^{-1} \boldsymbol{\Phi}_i, \qquad H'_i = H_i - \boldsymbol{\Phi}^*_i \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^{-1} \Omega(\boldsymbol{\Phi}, H), \\ &\mathfrak{C}'_i = \mathfrak{C}_i - \Omega(\mathfrak{C}, \boldsymbol{\Phi}^*) \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^{-1} \boldsymbol{\Phi}_i, \qquad \boldsymbol{x}' = \boldsymbol{x} - \Omega(\mathfrak{C}, \boldsymbol{\Phi}^*) \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^{-1} \Omega(\boldsymbol{\Phi}, H).\end{aligned} \tag{14}$$

Let us remark that the advantage of this particular fundamental transformations is that the transformation potentials are given explicitly by (8)–(10); i.e., they are obtained by an integration in the complex plane instead of by discrete integration of Eqs. (11)–(13). A very simple example appears when $m = 1$ and we take two diagonal spectral distributions

$$f(z) = \sum_{i=1}^{N} F_i(z) P_i, \qquad g(z) = \sum_{i=1}^{N} G_i(z) P_i,$$

where $\{F_i, G_i\}_{i=1}^N$ is a set of $2N$ scalar distributions on $\mathbb{C}$. The transformation potentials are

$$\begin{aligned}
&\Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*) = C + \mathrm{diag}(c_1, \dots, c_N),\\
&c_i := \int_{\mathbb{C}\times\mathbb{C}} F_i(z) \frac{1}{z - z'} \left[\frac{(z' - q_i)(z - p_i)}{(z' - p_i)(z - q_i)}\right]^{n_i} G_i(z')\, \mathrm{d}^2 z\, \mathrm{d}^2 z',\\
&\Omega(\boldsymbol{\Phi}, H) = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}, \qquad a_i := \int_{\mathbb{C}} F_i(z) \frac{1}{z} \left[\frac{q_i(z - p_i)}{p_i(z - q_i)}\right]^{n_i} \mathrm{d}^2 z,\\
&\Omega(\mathfrak{C}, \boldsymbol{\Phi}^*) = \mathrm{diag}(b_1, \dots, b_N), \qquad b_i := -\int_{\mathbb{C}} \frac{1}{z} \left[\frac{(z - q_i)p_i}{(z - p_i)q_i}\right]^{n_i} G_i(z)\, \mathrm{d}^2 z.
\end{aligned}$$

Thus, if $\Lambda = \Omega(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)^{-1}$, we can write for the points of the quadrilateral lattice

$$x_i(\boldsymbol{n}) = \frac{p_i - q_i}{p_i q_i} n_i - b_i(n_i) \sum_{j=1}^{N} \Lambda_{ij}(\boldsymbol{n}) a_j(n_j).$$

3. Reduced lattices

This section is devoted to explore which among the families of quadrilateral lattices presented above are of reduced type. We begin by introducing the particular Cartesian lattice to which proper fundamental transformations will be applied. Then, we present families of pseudo-circular lattices and of symmetric lattices. Finally, the quadrilateral lattices which are of both symmetric and pseudo-circular types are found and among them we isolate those of pseudo-Egorov type.

Cartesian lattice. For the reductions we need to consider the previous Cartesian background but with $q_i = -p_i$, $i = 1, \dots, N$. If this is the case, we have

$$\begin{aligned}
&Q_{ij}(\boldsymbol{n}) := 0, \qquad H_i(\boldsymbol{n}) := -2(-1)^{n_i - 1},\\
&\mathfrak{C}_i(\boldsymbol{n}) := \frac{1}{p_i}(-1)^{n_i} \boldsymbol{e}_i, \qquad \boldsymbol{x}(\boldsymbol{n}) := -2 \sum_{i=1}^{N} \frac{1}{p_i} n_i \boldsymbol{e}_i.
\end{aligned}$$

Solutions of the pseudo-circular lattice. We shall consider those quadrilateral lattices as given in (14) but with spectral data $\{f_k(z), g_k(z)\}_{k=1}^m$ satisfying

$$g_k(z) = -\frac{1}{2} z \sum_{l=1}^{m} f_l^\tau(-z) B_{kl}^\tau, \tag{15}$$

where B_{kl} are $N \times N$ matrices.

The notation

$$\omega_{kl}(\mathbf{\Phi}, \mathbf{\Phi}^*) := \frac{1}{2} \int_{\mathbb{C}\times\mathbb{C}} f_k(z) \frac{z'}{z+z'} \left(\sum_{i=1}^{N} \left(\frac{(z'-p_i)(z-p_i)}{(z'+p_i)(z+p_i)} \right)^{n_i} P_i \right) f_l^\tau(z')\, \mathrm{d}^2 z\, \mathrm{d}^2 z',$$
$$\omega_k(\mathfrak{C}, \mathbf{\Phi}^*) := \frac{1}{2} \int_{\mathbb{C}} f_k(z) \sum_{i=1}^{N} \left(\frac{p_i - z}{p_i + z} \right)^{n_i} \mathrm{d}^2 z\, P_i \tag{16}$$

allows us to write

$$\Omega_{kl}(\mathbf{\Phi}, \mathbf{\Phi}^*) = C_{kl} + \sum_{k'=1}^{m} \omega_{kk'}(\mathbf{\Phi}, \mathbf{\Phi}^*) B_{k'l}^\tau,$$
$$\Omega_k(\mathfrak{C}, \mathbf{\Phi}^*) = \left(\sum_{k'=1}^{m} B_{kk'} \omega_{k'}(\mathfrak{C}, \mathbf{\Phi}^*) \right)^\tau.$$

From (16), we deduce that

$$\omega_{kl}(\mathbf{\Phi}, \mathbf{\Phi}^*) + \omega_{lk}^\tau(\mathbf{\Phi}, \mathbf{\Phi}^*) = 2\omega_k(\mathfrak{C}, \mathbf{\Phi}^*) \omega_l^\tau(\mathfrak{C}, \mathbf{\Phi}^*), \tag{17}$$

and

$$(T_i + 1) \left(\frac{p_i - z}{p_i + z} \right)^{n_i} = \frac{p_i}{z + p_i} \left(\frac{p_i - z}{p_i + z} \right)^{n_i}$$

implies

$$((T_i + 1)\omega_k(\mathfrak{C}, \mathbf{\Phi}^*))^\tau \mathfrak{C}_i = \mathbf{\Phi}_i. \tag{18}$$

From (17) and (18), we can conclude that when

$$\sum_{k'=1}^{N} (B_{kk'} C_{k'l} + C_{k'k}^\tau B_{lk'}^\tau) = 0$$

is verified, the following equations hold:

$$B\Omega(\mathbf{\Phi}, \mathbf{\Phi}^*) + \Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)^\tau B^\tau = 2\Omega(\mathfrak{C}, \mathbf{\Phi}^*)^\tau \Omega(\mathfrak{C}, \mathbf{\Phi}^*),$$
$$((T_i + 1)\Omega(\mathfrak{C}, \mathbf{\Phi}^*))^\tau \mathfrak{C}_i = B\mathbf{\Phi}_i,$$

where we use the notation $B := (B_{kl})$. Moreover, from the equation

$$((T_i - 1)\Omega(\mathfrak{C}, \mathbf{\Phi}^*))^\tau \mathfrak{C}_i = I_{p,q}(T_i \mathbf{\Phi}_i^*)^{\mathrm{t}} \|\mathfrak{C}_i\|^2 = \frac{\epsilon_i}{p_i^2} I_{p,q}(T_i \mathbf{\Phi}_i^*)^{\mathrm{t}},$$

it follows that

$$\Omega(\mathfrak{C}, \mathbf{\Phi}^*)^\tau \mathfrak{C}_i = \frac{1}{2}\left(B\mathbf{\Phi}_i - \frac{\epsilon_i}{p_i^2} I_{p,q}(T_i \mathbf{\Phi}_i^*)^{\mathrm{t}}\right),$$

$$(T_i \Omega(\mathfrak{C}, \mathbf{\Phi}^*))^\tau \mathfrak{C}_i = \frac{1}{2}\left(B\mathbf{\Phi}_i - \frac{\epsilon_i}{p_i^2} I_{p,q}(T_i \mathbf{\Phi}_i^*)^{\mathrm{t}}\right),$$

Following [11,13] the associated fundamental transformation with such data gives the following pseudo-circular lattice:

$$\boldsymbol{x}' = \boldsymbol{x} - \omega^\tau(\mathfrak{C}, \mathbf{\Phi}^*) B^\tau (C + \omega(\mathbf{\Phi}, \mathbf{\Phi}^*) B^\tau)^{-1} \Omega(\mathbf{\Phi}, H).$$

The first potentials ρ_i' of new lattice [8] are

$$\rho_i' = \|\mathfrak{C}_i\|^2 = \frac{\epsilon_i}{p_i^2}(1 + (T_i \mathbf{\Phi}_i^*)\Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)^{-1} \mathbf{\Phi}_i). \tag{19}$$

Let us consider a simple example with $m = 1$ and a diagonal spectral distribution

$$f(z) = \sum_{i=1}^{N} F_i(z) P_i,$$

where $\{F_i\}_{i=1}^N$ is a set of N scalar distributions on $\mathbb{C}$. Then, if we denote

$$a_i := \int_{\mathbb{C}} F_i(z) \frac{1}{z}\left[\frac{p_i - z}{p_i + z}\right]^{n_i} \mathrm{d}^2 z, \qquad b_i := \int_{\mathbb{C}} \left[\frac{p_i - z}{p_i + z}\right]^{n_i} F_i(z)\, \mathrm{d}^2 z,$$

the transformation potentials are

$$\Omega(\mathbf{\Phi}, \mathbf{\Phi}^*) = C + \frac{1}{4}\mathrm{diag}(b_1^2, \dots, b_N^2) B^\tau,$$

$$\Omega(\mathbf{\Phi}, H) = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}, \qquad \Omega(\mathfrak{C}, \mathbf{\Phi}^*) = \frac{1}{2}\mathrm{diag}(b_1, \dots, b_N) B^\tau,$$

and a new pseudo-circular lattice is given by

$$x_i(\boldsymbol{n}) = -\frac{2}{p_i} n_i - \frac{1}{2} b_i(n_i) B^\tau \left(C + \frac{1}{4}\mathrm{diag}(b_1^2(n_1), \dots, b_N^2(n_N)) B^\tau\right)^{-1} \begin{pmatrix} a_1(n_1) \\ \vdots \\ a_N(n_N) \end{pmatrix},$$

whenever

$$BC+(BC)^{\tau}=0.$$

Nice examples of this construction appear as follows. Let $\gamma=\mathrm{i}\mathbb{R}$ be the imaginary axis in the complex plane and let us take an spectral measure concentrated over γ

$$\frac{F_i(z)}{z}=\mathcal{F}_i(y)\delta_\gamma(z),\quad i=1,\dots,N,\quad y\in\mathbb{R}.$$

Then, observing that

$$\frac{p_i-\mathrm{i}y}{p_i+\mathrm{i}y}=\exp\left(-2\mathrm{i}\arctan\left(\frac{y}{p_i}\right)\right),$$

we get

$$a_i(n_i)=\int_{\mathbb{R}}\mathcal{F}_i(y)\exp\left(-2\mathrm{i}\arctan\left(\frac{y}{p_i}\right)n_i\right)\,\mathrm{d}y,$$
$$b_i(n_i)=\int_{\mathbb{R}}\mathrm{i}y\mathcal{F}_i(y)\exp\left(-2\mathrm{i}\arctan\left(\frac{y}{p_i}\right)n_i\right)\,\mathrm{d}y.$$

Performing the change of variables

$$\phi=\arctan\left(\frac{y}{p_i}\right),$$

one gets

$$a_i(n_i)=\int_{-\pi/2}^{\pi/2}\mathcal{F}_i(p_i\tan\phi)\exp(-2\mathrm{i}n_i\phi)\frac{p_i}{\cos^2\phi}\,\mathrm{d}\phi,$$
$$b_i(n_i)=\int_{-\pi/2}^{\pi/2}\mathrm{i}p_i\tan(\phi)\mathcal{F}_i(p_i\tan\phi)\exp(-2\mathrm{i}n_i\phi)\frac{p_i}{\cos^2\phi}\,\mathrm{d}\phi.$$

In terms of

$$\tilde{\mathcal{F}}_i(\phi):=p_i\frac{\mathcal{F}_i(p_i\tan\phi)}{\cos^4\phi},$$

we have

$$\begin{aligned}a_i(n_i)&=\int_{-\pi/2}^{\pi/2}\tilde{\mathcal{F}}_i(\phi)\exp(-2\mathrm{i}n_i\phi)\cos^2\phi\,\mathrm{d}\phi\\&=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\tilde{\mathcal{F}}_i(\phi)\exp(-2\mathrm{i}n_i\phi)(1+\cos 2\phi)\,\mathrm{d}\phi,\\b_i(n_i)&=\int_{-\pi/2}^{\pi/2}\mathrm{i}p_i\tilde{\mathcal{F}}_i(\phi)\exp(-2\mathrm{i}n_i\phi)\sin\phi\cos\phi\,\mathrm{d}\phi\\&=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\mathrm{i}p_i\tilde{\mathcal{F}}_i(\phi)\exp(-2\mathrm{i}n_i\phi)\sin 2\phi\,\mathrm{d}\phi.\end{aligned}$$

Finally, if we define the Fourier coefficients of $\tilde{\mathcal{F}}_i$ as

$$\gamma_i(n_i) = \int_{-\pi/2}^{\pi/2} \tilde{\mathcal{F}}_i(\phi)\exp(-\mathrm{i}2n_i\phi)\,\mathrm{d}\phi, \quad n \in \mathbb{Z},$$

we have

$$a_i(n_i) = \tfrac{1}{4}(2 + T_i^{-1} + T_i)\gamma_i(n_i), \qquad b_i(n_i) = p_i\tfrac{1}{4}(T_i^{-1} + T_i)\gamma_i(n_i).$$

Thus, all the elements of the new pseudo-circular lattice are built up in terms of the functions $\gamma_i, i = 1, \ldots, N$. For example, if $\{\alpha_s\}_{s\in S} \subset [-\pi/2, \pi/2]$ and $\tilde{\mathcal{F}}_i(\phi) = (1/2)\sum_{s\in S} A_{is}$ $(\delta(\phi - \alpha_{is}) + \delta(\phi + \alpha_{is}))$, we get

$$\gamma_i(n_i) = \sum_{s\in S} A_{is}\cos 2n_i\alpha_{is}, \quad n_i \in \mathbb{Z}.$$

Observe that, a priori, $\alpha_{is} \in [-\pi/2, \pi/2]$, however, if we add to α_{is} any integer multiple of π the function γ_i does not change. Then, the coefficients are not constrained to belong to $[-\pi/2, \pi/2]$ and we can write $\alpha_{is} \in \mathbb{R}$. Therefore, the functions can be taken as any cosine series, for example the one given by the Jacobi elliptic cosine

$$\mathrm{cn}(u|m) := \frac{2\pi}{\sqrt{m}K}\sum_{s=0}^{\infty}\frac{q^{s+1/2}}{1 + q^{2s+1}}\cos\left(\frac{(2s+1)\pi}{2K}u\right)$$

with argument m, real and imaginary quarter periods K and K'

$$K = \int_0^{\pi/2}(1 - m\sin^2\theta)^{-1/2}\,\mathrm{d}\theta, \qquad K' = \int_0^{\pi/2}(1 - (1 - m)\sin^2\theta)^{-1/2}\,\mathrm{d}\theta,$$

and nome $q = \exp(-\pi K'/K)$.

Hence, an elliptic example with

$$\gamma_i(n_i) = A_i\,\mathrm{cn}(n_i|m_i), \quad A_i \in \mathbb{R}$$

is

$$a_i(n_i) = \tfrac{1}{4}A_i(2\mathrm{cn}(n_i|m_i) + \mathrm{cn}(n_i - 1|m_i) + \mathrm{cn}(n_i + 1|m_i)),$$
$$b_i(n_i) = p_i\tfrac{1}{4}A_i(\mathrm{cn}(n_i - 1|m_i) - \mathrm{cn}(n_i + 1|m_i)).$$

It is of interest to study the relation of these solutions with the quasi-periodic circular lattices found by P. Grinevich (private communication of A. Doliwa).

Finally, we consider two particular examples of circular lattices in $\mathbb{R}^2$ and circular discrete surfaces in $\mathbb{R}^3$. In the first place, we take $p_1 = p_2 = -2$, $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$\gamma_i(n) := 2\mathrm{cn}(\tfrac{7}{10}n|\tfrac{1}{2}), \quad i = 1, 2,$$

and the 2D circular lattice is shown in Fig. 1.

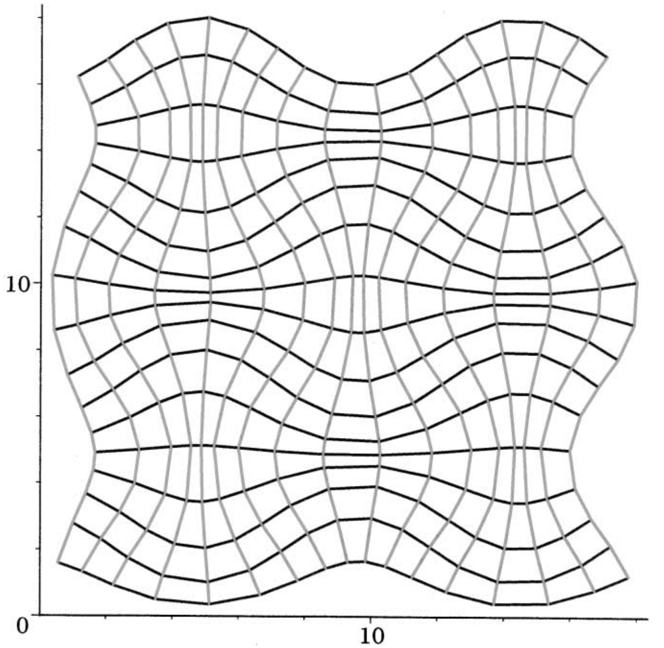

Fig. 1.

We now plot a circular discrete surface, with $p_1 = p_2 = p_3 = -2$. $C = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$ and

$$\gamma_i(n) := \tfrac{1}{10}\mathrm{cn}(\tfrac{1}{2}n|\tfrac{1}{2}), \quad i = 1, 2, 3.$$

The corresponding discrete surface for $n_3 = 0$ is shown in Fig. 2.

3.1. Solutions of the symmetric lattice

We next study quadrilateral lattices as given by (14) but with spectral data $\{f_k(z), g_k(z)\}_{k=1}^m$ satisfying

$$g_k(z) = -\frac{1}{2}\sum_{l=1}^{m} f_l^\tau(-z)B_{kl}^\tau, \tag{20}$$

where B_{kl} are $N \times N$ matrices.

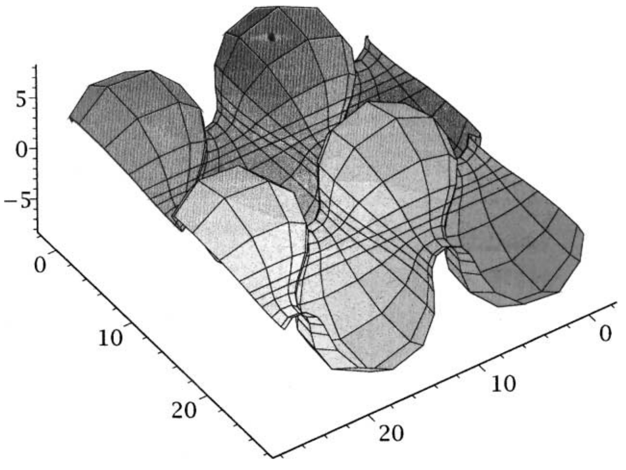

Fig. 2.

We introduce the notation

$$\omega_{kl}(\mathbf{\Phi}, \mathbf{\Phi}^*) := -\frac{1}{2}\int_{\mathbb{C}\times\mathbb{C}} f_k(z)\frac{1}{z+z'}\left(\sum_{i=1}^{N}\left(\frac{(z'-p_i)(z-p_i)}{(z'+p_i)(z+p_i)}\right)^{n_i} P_i\right) f_l^{\tau}(z')\,\mathrm{d}^2 z\,\mathrm{d}^2 z',$$

$$\omega_k(\mathfrak{C}, \mathbf{\Phi}^*) := -\frac{1}{2}\int_{\mathbb{C}} f_k(z)\frac{1}{z}\sum_{i=1}^{N}\left(\frac{p_i-z}{p_i+z}\right)^{n_i}\,\mathrm{d}^2 z\,P_i,$$

so that

$$\Omega_{kl}(\mathbf{\Phi}, \mathbf{\Phi}^*) = C_{kl} + \sum_{k'=1}^{m}\omega_{kk'}(\mathbf{\Phi}, \mathbf{\Phi}^*)B^{\tau}_{k'l},$$

$$\Omega_k(\mathfrak{C}, \mathbf{\Phi}^*) = \left(\sum_{k'=1}^{m} B_{kk'}\omega_{k'}(\mathfrak{C}, \mathbf{\Phi}^*)\right)^{\tau}.$$

It is not difficult to realize that

$$\omega_{kl}(\mathbf{\Phi}, \mathbf{\Phi}^*) = \omega^{\tau}_{lk}(\mathbf{\Phi}, \mathbf{\Phi}^*). \tag{21}$$

If

$$\sum_{k'=1}^{N}(B_{kk'}C_{k'l} - C^{\tau}_{k'k}B^{\tau}_{lk'}) = 0,$$

one concludes that

$$I_{p,q}B\Omega(\mathbf{\Phi}, \mathbf{\Phi}^*) = \Omega^{\mathrm{t}}(\mathbf{\Phi}, \mathbf{\Phi}^*)B^{\mathrm{t}}I_{p,q}, \qquad T_i\mathbf{\Phi}^*_i = \frac{\epsilon_i}{p_i}\mathbf{\Phi}^{\mathrm{t}}_i B^{\mathrm{t}} I_{p,q},$$

which, according to [13], ensures that the associated fundamental transformation preserves the symmetric character of the lattice.

A simple example appears for $m=1$ and

$$f(z)=\sum_{i=1}^{N}F_i(z)P_i,$$

where $\{F_i\}_{i=1}^N$ is a set of N scalar distributions on $\mathbb{C}$. With the notation

$$a_i:=\int_{\mathbb{C}}F_i(z)\frac{1}{z}\left[\frac{p_i-z}{p_i+z}\right]^{n_i}\,\mathrm{d}^2z,$$
$$b_i:=\int_{\mathbb{C}\times\mathbb{C}}\frac{F_i(z)F_i(z')}{z+z'}\left(\frac{(z'-p_i)(z-p_i)}{(z'+p_i)(z+p_i)}\right)^{n_i}\,\mathrm{d}^2z\,\mathrm{d}^2z',$$

the transformation potentials are

$$\Omega(\boldsymbol{\Phi},\boldsymbol{\Phi}^*)=C-\frac{1}{2}\mathrm{diag}(b_1,\dots,b_N)B^\tau,$$
$$\Omega(\boldsymbol{\Phi},H)=\begin{pmatrix}a_1\\ \vdots\\ a_N\end{pmatrix},\qquad \Omega(\mathfrak{C},\boldsymbol{\Phi}^*)=-\frac{1}{2}\mathrm{diag}(a_1,\dots,a_N)B^\tau.$$

Thus, a new symmetric lattice is given by

$$x_i(\boldsymbol{n})=-\frac{2}{p_i}n_i+\frac{1}{2}a_i(n_i)B^\tau\left(C-\frac{1}{2}\mathrm{diag}(b_1(n_1),\dots,b_N(n_N))B^\tau\right)^{-1}\begin{pmatrix}a_1(n_1)\\ \vdots\\ a_N(n_N)\end{pmatrix},$$

whenever

$$BC=(BC)^\tau.$$

3.2. Pseudo-Egorov lattices

To construct symmetric pseudo-circular lattices set $m=2M$, and

$$f_k(z)=\delta(z-\mu_k),\qquad f_{k+M}(z)=\delta(z+\mu_k),\quad k=1,\dots,M,\qquad C=I_{2NM},$$

where μ_k are non-zero complex numbers different from p_i and such that $\mu_k+\mu_l\neq 0$.

For the construction of a pseudo-circular lattice, we take the matrix B_{pc} in the form

$$B_{\mathrm{pc}}=\begin{pmatrix}0 & -J_{\mathrm{pc}}\\ J_{\mathrm{pc}}^\tau & 0\end{pmatrix},$$
$$J_{\mathrm{pc}}:=-2\,\mathrm{diag}\left(\frac{B_1}{\mu_1},\dots,\frac{B_M}{\mu_M}\right),\quad J_{\mathrm{pc}}^\tau:=-2\,\mathrm{diag}\left(\frac{B_1^\tau}{\mu_1},\dots,\frac{B_M^\tau}{\mu_M}\right),$$

where B_j are some $N \times N$ matrices. Instead, for the construction of a symmetric lattice we take the matrix B_{s} as

$$B_{\mathrm{s}} = \begin{pmatrix} 0 & J_{\mathrm{s}} \\ J_{\mathrm{s}}^{\tau} & 0 \end{pmatrix}, \quad J_{\mathrm{s}} := -2\,\mathrm{diag}(B_1, \dots, B_M), \quad J_{\mathrm{s}}^{\tau} := -2\,\mathrm{diag}(B_1^{\tau}, \dots, B_M^{\tau}).$$

It is easily checked that these two lattices are the same due to the fact that the distributions g_k coincide in both prescriptions, namely

$$g_k(z) = \delta(z - \mu_k)B_k, \qquad g_{k+M} = \delta(z + \mu_k)B_k^{\tau}, \quad k = 1, \dots, M.$$

Thus, we have constructed a pseudo-circular symmetric lattice. However, this construction has a pathology which is only cured if some further requirements are imposed over the matrices B_k. The problem is the appearance of singularities in $\Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)$ as we have

$$\Omega_{k,l}(\mathbf{\Phi}, \mathbf{\Phi}^*) = \delta_{kl} + D(\mu_k, \mu_l)B_l, \qquad \Omega_{k,l+M}(\mathbf{\Phi}, \mathbf{\Phi}^*) = D(\mu_k, -\mu_l)B_l^{\tau},$$
$$\Omega_{l+M,k}(\mathbf{\Phi}, \mathbf{\Phi}^*) = D(-\mu_k, \mu_l)B_l, \qquad \Omega_{k+M,l+M}(\mathbf{\Phi}, \mathbf{\Phi}^*) = \delta_{kl}+D(-\mu_k, -\mu_l)B_l^{\tau},$$

where $k, l = 1, \dots, M$. The matrices $\Omega_{k,k}(\mathbf{\Phi}, \mathbf{\Phi}^*)$ and $\Omega_{k+M,k+M}(\mathbf{\Phi}, \mathbf{\Phi}^*)$, $k = 1, \dots, M$ are singular. A method to avoid this drawback is to use the matrices B_k in such a way that the singularity is killed. Observe that these singularities appear in the expressions for the new rotation and Lamé coefficients, tangent vectors and points of the lattice in terms of $B_k\Omega_{k,k}(\mathbf{\Phi}, \mathbf{\Phi}^*)^{-1}$ and $B_k^{\tau}\Omega_{k+M,k+M}(\mathbf{\Phi}, \mathbf{\Phi}^*)^{-1}$. Now, we can proceed as in [16]: take B_k nilpotent matrices; i.e. $B_k^2 = 0$. To check that the suggested mechanism works properly, we first consider a quadrilateral lattice generated by the same f's but with the g's as follows:

$$g_k(z) = \delta(z - \nu_k)B_k, \qquad g_{k+M} = \delta(z + \nu_k)B_k^{\tau}, \quad k = 1, \dots, M,$$

and with the ν_k, $k = 1, \dots, M$, being arbitrary complex numbers. Only when $\nu_k \to \mu_k$, we can ensure that the resulting lattice is symmetric and pseudo-circular; i.e., we shall consider this limit carefully. In particular, let us deal with

$$B_k\Omega_{kk}(\mathbf{\Phi}, \mathbf{\Phi}^*)^{-1} = \lim_{\nu_k \to \mu_k} B_k(I_N + D(\mu_k, \nu_k)B_k)^{-1}.$$

The behaviour

$$D(\mu_k, \nu_k) = \frac{1}{\mu_k - \nu_k} + d(\mu_k) + \mathcal{O}(\mu_k - \nu_k) \quad \text{as } \nu_k \to \mu_k,$$

where

$$d(z) := 2\sum_{i=1}^{N} \frac{p_i}{(z - p_i)(z + p_i)} P_i,$$

allows us to write

$$B_k(I_N+D(\mu_k, \nu_k)B_k)^{-1} \sim B_k\left(\frac{B_k}{\mu_k-\nu_k}+(I_N+d(\mu_k)B_k)+\mathcal{O}(\mu_k-\nu_k)B_k\right)^{-1},$$
$$\nu_k \to \mu_k.$$

When we assume that $I_N + d(\mu_k)B_k$ is invertible the nilpotency of B_k ensures that $(I_N + d(\mu_k)B_k)^{-1}B_k = B_k$ and therefore $B_k(I_N + d(\mu_k)B_k)^{-1}B_k = 0$. Thus,

$$B_k(I_N + d(\mu_k)B_k)^{-1}\left(\frac{B_k}{\mu_k - \nu_k} + (I_N + d(\mu_k)B_k) + \mathcal{O}(\mu_k - \nu_k)B_k\right)$$
$$= B_k + B_k\mathcal{O}(\mu_k - \nu_k)B_k,$$

which implies

$$B_k\Omega_{kk}(\mathbf{\Phi}, \mathbf{\Phi}^*) = B_k(I_N + d(\mu_k)B_k)^{-1}.$$

Then, we get a symmetric pseudo-circular lattice if we substitute the ill-defined transformation potentials $\Omega_{k,k}(\mathbf{\Phi}, \mathbf{\Phi}^*)$ and $\Omega_{k+M,k+M}(\mathbf{\Phi}, \mathbf{\Phi}^*)$ by

$$\Omega_{k,k}(\mathbf{\Phi}, \mathbf{\Phi}^*) \to I_N + d(\mu_k)B_k, \qquad \Omega_{k+M,k+M}(\mathbf{\Phi}, \mathbf{\Phi}^*) \to I_N + d(-\mu_k)B_k,$$

respectively.

As it was proved in [8], a symmetric pseudo-circular lattice is of Egorov type iff the potentials of the symmetric lattice coincide with the potentials of the pseudo-circular lattice, which in our case happens only when

$$p_i = p, \quad i = 1, \dots, N.$$

To illustrate what type of solutions we get, let us study the case where

$$B = \sum_{i=1,\dots,r;k=r+1,\dots,N} b_{ik}E_{ik}, \quad 1 \le r \le N.$$

This form of the matrix B ensures that $BDB = 0$ for any diagonal matrix. In particular,

$$B(I_N + d(\mu)B)^{-1} = B.$$

We now introduce the functions

$$E(z) := \sum_{i=1}^{N}\left(\frac{p_i - z}{p_i + z}\right)^{n_i} P_i, \qquad E(\mu) := \begin{pmatrix} \left(\frac{p_1 - z}{p_1 + z}\right)^{n_1} \\ \vdots \\ \left(\frac{p_N - z}{p_N + z}\right)^{n_N} \end{pmatrix},$$

and observe that

$$E(-z) = E(z)^{-1}, \qquad D(z, z') = \frac{1}{z - z'}E(z)E(-z').$$

Hence, our effective transformation potentials are

$$\Omega(\mathbf{\Phi}, \mathbf{\Phi}^*) = \begin{pmatrix} I_N & \frac{1}{2\mu}E(\mu)^2 B^\tau \\ -\frac{1}{2\mu}E(-\mu)^2 B & I_N \end{pmatrix},$$

$$\Omega(\mathfrak{C}, \mathbf{\Phi}^*) = \frac{1}{\mu}(-E(-\mu)B, E(\mu)B^\tau), \qquad \Omega(\mathbf{\Phi}, H) = \frac{1}{\mu}\begin{pmatrix} E(\mu) \\ -E(-\mu) \end{pmatrix},$$

and we have the expression

$$\Omega(\mathbf{\Phi}, \mathbf{\Phi}^*)^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with

$$\alpha := \left(I_N + \frac{1}{4\mu^2}E(\mu)^2 B^\tau E(-\mu)^2 B\right)^{-1},$$
$$\beta := -\frac{1}{2\mu}\left(I_N + \frac{1}{4\mu^2}E(\mu)^2 B^\tau E(-\mu)^2 B\right)^{-1} E(\mu)^2 B^\tau,$$
$$\gamma := \frac{1}{2\mu}\left(I_N + \frac{1}{4\mu^2}E(-\mu)^2 B E(\mu)^2 B^\tau\right)^{-1} E(-\mu)^2 B,$$
$$\delta := \left(I_N + \frac{1}{4\mu^2}E(-\mu)^2 B E(\mu)^2 B^\tau\right)^{-1}.$$

Therefore, a symmetric pseudo-circular lattice is

$$\boldsymbol{x}' = -2\begin{pmatrix} \frac{n_1}{p_1} \\ \vdots \\ \frac{n_N}{p_N} \end{pmatrix} - [-E(-\mu)B\alpha + E(\mu)B^\tau\gamma]\frac{1}{\mu^2}E(\mu)$$
$$+[-E(-\mu)B\beta + E(\mu)B^\tau\delta]\frac{1}{\mu^2}E(-\mu).$$

A more closed form of this solution is found if we introduce the notation

$$M := \sum_{i,j=1}^{r} m_{ij}E_{ij}, \quad m_{ij} := \frac{1}{4\mu^2}\left(\frac{p_i+\mu}{p_i-\mu}\right)^{2n_i} \sum_{k=r+1}^{N}\left(\frac{p_k-\mu}{p_k+\mu}\right)^{2n_k} \epsilon_j\epsilon_k b_{ik}b_{jk},$$
$$\tilde{M} = \sum_{i,j=1}^{r} \tilde{m}_{ij}E_{ij}, \qquad \sum_{i'=1}^{r}\tilde{m}_{ii'}(\delta_{i'j} + m_{i'j}) = \delta_{ij},$$

so that

$$B\alpha = E(\mu)^2 \tilde{M} E(-\mu)^2 B, \qquad B\beta = -2\mu E(\mu)^2 \left(\sum_{i=1}^{r} E_{ii} - \tilde{M}\right),$$
$$B^\tau \gamma = \frac{1}{2\mu} B^\tau \tilde{M} E(-\mu)^2 B, \qquad B^\tau \delta = B^\tau \tilde{M},$$

and the final expression for the symmetric pseudo-circular lattice becomes

$$\boldsymbol{x}' = -2 \begin{pmatrix} \dfrac{n_1}{p_1} \\ \vdots \\ \dfrac{n_N}{p_N} \end{pmatrix} - \left[-E(\mu)\tilde{M}E(-\mu)^2 B + \frac{1}{2\mu} E(\mu) B^\tau \tilde{M} E(-\mu)^2 B\right] \frac{1}{\mu^2} E(\mu)$$
$$+ \left[2\mu E(\mu)\left(\sum_{i=1}^{r} E_{ii} - \tilde{M}\right) + E(\mu) B^\tau \tilde{M}\right] \frac{1}{\mu^2} E(-\mu).$$

A further simplification arises by assuming $r = 1$ so that

$$B = \sum_{k=2}^{N} b_k E_{1k}, \qquad B^\tau = \epsilon_1 \sum_{k=2}^{N} b_k \epsilon_k E_{k1}, \qquad M = mE_{11},$$
$$m = \frac{1}{4\mu^2}\epsilon_1 \sum_{k=2}^{N} \epsilon_k b_k^2 \left(\frac{p_k - \mu}{p_k + \mu}\right)^{2n_k} \left(\frac{p_1 - \mu}{p_1 + \mu}\right)^{-2n_1}, \qquad \tilde{M} = \tilde{m} E_{11},$$
$$\tilde{m} = \frac{1}{1+[(1/4\mu^2)\epsilon_1 \sum_{k=2}^{N} \epsilon_k b_k^2((p_k-\mu)/(p_k+\mu))^{2n_k}((p_1 - \mu)/(p_1 + \mu))^{-2n_1}]}.$$

The following formulae characterize an N-dimensional symmetric pseudo-circular lattice

$$x_1 = -2\frac{n_1}{p_1} + \frac{2}{\mu}(1 - \tilde{m}) + \frac{1}{\mu^2}\tilde{m} \sum_{k=2}^{N} b_k \left(\frac{p_k - \mu}{p_k + \mu}\right)^{n_k} \left(\frac{p_1 - \mu}{p_1 + \mu}\right)^{-n_1},$$
$$x_k = -2\frac{n_k}{p_k} + \frac{1}{\mu^2}\epsilon_1 \tilde{m} \epsilon_k b_k \left(\frac{p_k - \mu}{p_k + \mu}\right)^{n_k} \left(\frac{p_1 - \mu}{p_1 + \mu}\right)^{-n_1}$$
$$\times \left(1 - \frac{1}{2\mu} \sum_{l=2}^{N} b_l \left(\frac{p_l - \mu}{p_l + \mu}\right)^{n_l} \left(\frac{p_1 - \mu}{p_1 + \mu}\right)^{n_1}\right),$$

where $k = 2, \ldots, N$. Only when $p_i = p, i = 1, \ldots, N$ this symmetric pseudo-circular

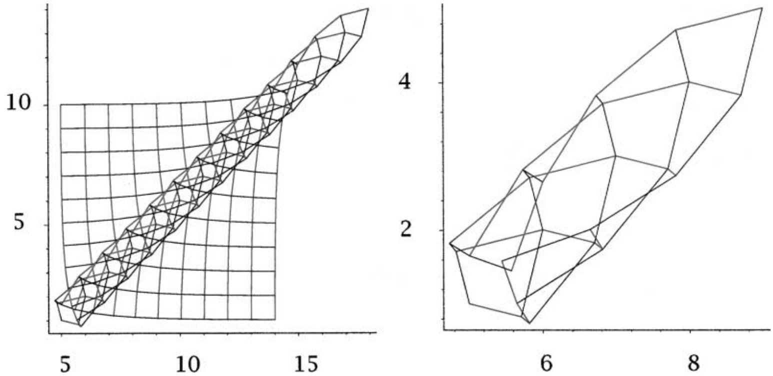

Fig. 3.

lattice is a pseudo-Egorov lattice with

$$x_1 = -2\frac{n_1}{p} + \frac{2}{\mu}(1-\tilde{m}) + \frac{1}{\mu^2}\tilde{m}\sum_{k=2}^{N} b_k \left(\frac{p-\mu}{p+\mu}\right)^{n_k-n_1},$$

$$x_k = -2\frac{n_k}{p} + \frac{1}{\mu^2}\epsilon_1\tilde{m}\epsilon_k b_k \left(\frac{p-\mu}{p+\mu}\right)^{n_k-n_1}\left(1-\frac{1}{2\mu}\sum_{l=2}^{N} b_l \left(\frac{p-\mu}{p+\mu}\right)^{n_l-n_1}\right),$$

$$\tilde{m} = \frac{1}{1+[(1/4\mu^2)\epsilon_1\sum_{k=2}^{N}\epsilon_k b_k^2((p-\mu)/(p+\mu))^{2(n_k-n_1)}]}.$$

Finally, we shall represent explicit examples in two and three dimensions. We take $p = -2$, $\mu = 1/2$, and $b_i = 1$.

The picture on the left of Fig. 3 is an overview of the lattice, we see that on the diagonal there is a mess of lines. This phenomena is connected with the Egorov nets presented in [16], where the net was not regular on the diagonal and, in fact, one has an Egorov atlas with two charts. The picture on the right of Fig. 3 is a detailed view of the lattice in the neighbourhood of the diagonal. We see that together with convex quadrilaterals one has also skew quadrilaterals, those quadrilaterals cross with the previous convex quadrilaterals. This is the discrete version of the mentioned non-regularity.

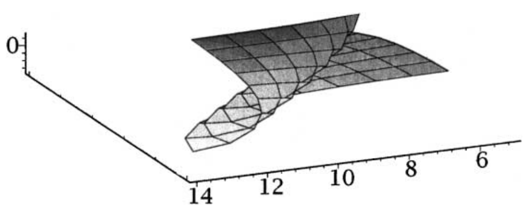

Fig. 4.

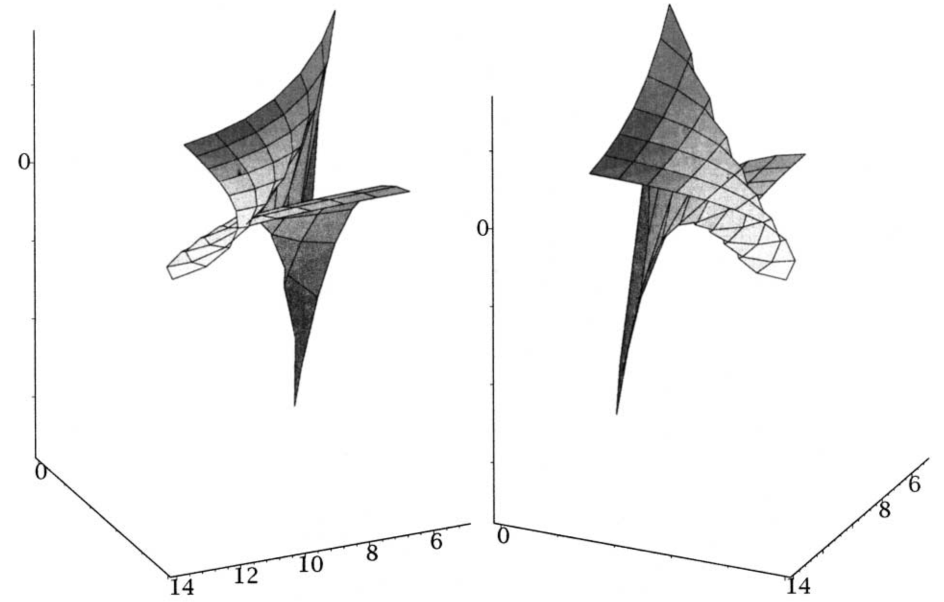

Fig. 5.

In $\mathbb{R}^3$, however, the mess in the discrete surfaces disappears as it is lifted up (Fig. 4). Next, we show three plots (Figs. 4 and 5) of a typical view of our discrete Egorov surfaces ($n_3 = 0$), observe that the two last plots (Fig. 5), right and left view of the surface, have been scaled vertically in order to better show the behaviour of the discrete Egorov surface. Observe that almost all the quadrilaterals are convex, however, there are two rows of skew quadrilaterals, that in the continuous limit form the lines of non-regularity of the surface.

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